Octonions

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1 History

The octonions are essentially an eight-dimensional analogue to the complex numbers. They came about as a result of exploration into the idea of square roots of negative numbers, which then developed into the idea of using multiple imaginary axes, going beyond just i.

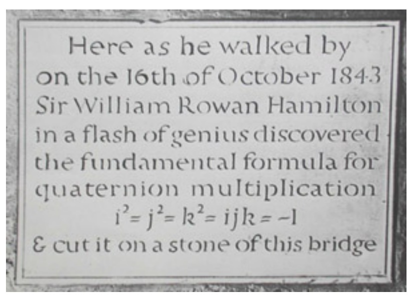
The first recorded usage of complex numbers is in Girolamo Cardano's work, *Ars Magna* (The Great Art), published in 1545. He was working with various problems in algebra, and at one point he needed to use $\sqrt{-15}$ to solve the problem "To divide 10 in two parts, the product of which is 40" – the solution of which is $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. [3, pp. 2–3]

This had mathematicians pondering over square roots of negative numbers, an entirely new type of number that could not be expressed using the reals. In his 1637 paper *La Geometrie*, Rene Descartes called these new quantities the imaginary numbers [4, p. 6], and Leonard Euler standardised the notation i for $\sqrt{-1}$. [5, p. 88]

The newfound language and notation prompted mathematicians to begin working with complex numbers in the form $a + bi$ – for $a, b \in \mathbb{R}$ – and try to find ways of understanding them and their uses. The rules of complex arithmetic were first formalised by Rafael Bombelli in 1572 [6, p. 5], and then in the early 1800s, Jean-Robert Argand popularised the idea of describing complex numbers as points in a plane, and thinking about operations (addition, subtraction, multiplication, division) as maps on the plane. [7, p. 2]

In 1835, William Rowan Hamilton decided to take a different approach: he started thinking about treating complex numbers $a + bi$ as ordered pairs (a, b). He discovered the rules necessary for operations between complex numbers as ordered pairs of real numbers, but he suspected that you could take this idea further, and tried looking into ordered triplets (a, b, c), especially multiplying them together with a rule that allowed for division by any non-zero element. He was aiming to invent a three-dimensional algebra that could describe operations in three-dimensional geometry, in the same way that complex numbers worked with two-dimensional geometry, as shown by Argand. What he was attempting was mathematically impossible in three dimensions; however, it was possible with four. [7, p. 3]

It is said that he was walking along the Royal Canal in Dublin when he had this epiphany, and carved a single line of calculation under Broome Bridge, now commemorated by a plaque (photo from *Hamilton Year 2005*):



$$i^2 = j^2 = k^2 = ijk = -1 \text{ [1, p. 145]}$$

Using these three linearly independent imaginaries i, j, k, he defined the set of quaternions, $\mathbb{H}$, which can be thought of as

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

analogously to $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

With a bit of manipulation, and taking 1 as the multiplicative identity, you can use the line of working above to come up with all of the rules for multiplying two distinct elements out of $\{1, i, j, k\}$, which in turn gives you all of the information you need to multiply any two quaternions together.

Hamilton then told his friend, John Graves, about the quaternions, who asked if you could just keep creating more “imaginaries”. Graves later sent him a letter saying that he had come up with a number system called the "octaves" (later octonions), which allowed you to multiply 8-tuples of real numbers in a way that also allowed for division, using seven linearly independent square roots of -1. Though the octonions were discovered in 1843 by Graves, Arthur Cayley beat him to publication after independently discovering them, calling them Cayley numbers. [[7, p. 5]; [2, pp. 8–9]]

After this discovery, the octonions faded into obscurity, being somewhat ahead of their time – people were still struggling with the idea of four-dimensional spaces[1], let alone eight. With higher dimensions unexplored, nobody knew what to use this number system for. Even now, they are nowhere near as well-known as their predecessors, the complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$, since their main uses come up when working in seven and eight dimensions. Despite this, they are still interesting as a glance into extending familiar number systems, as one of the lesser-known results of a centuries-long journey into the world of imaginary numbers, and, more pragmatically, for understanding higher-dimensional analogues of familiar operations and theorems such as the three-dimensional cross product and N-square identities.

2 Groundwork

2.1 Definitions

Definition 1 (Octonions [[9, p. 237],[1, p. 150]])**.** The set of octonions, denoted $\mathbb{O}$, is a number system that extends the quaternions, in the same way as quaternions extend the complex numbers, and the complex numbers extend the reals. It forms an eight-dimensional vector space over $\mathbb{R}$, with the standard basis $\{1, e_1, \ldots, e_7\}$.

Using the same ideas as before for defining $\mathbb{C}$ and $\mathbb{H}$, we can write

$$\mathbb{O} = \{a_0 + a_1e_1 + \cdots + a_7e_7 : a_0, \ldots, a_7 \in \mathbb{R}\}$$

Remark. Since octonions aren't used very often, the conventions are fairly loose – for example, the standard basis is written by John Conway and Derek Smith as $\{1, i_0, \ldots, i_6\}$ [2, p. 65] and by Tevian Dray and Corinne Manogue[2] as $\{1, i, j, k, kl, jl, il, l\}$ [6, p. 14]. I've chosen to stick with $\{1, e_1, \ldots, e_7\}$.

The above definition considers the octonions as a vector space, but in order to work with them properly, we need to give them more structure, considering $\mathbb{O}$ as a real normed division algebra.

Definition 2 (Algebra [1, p. 149])**.** An algebra is a vector space A over a field K combined with a bilinear product – i.e. not only are operations for addition of vectors and scalar multiplication defined, but you can also multiply vectors together, and this is distributive over addition and scalar multiplication.

[1]This much is evident from attempted explanations from around the time (see [8, pp. 4-6])!

[2]The notation used in their book came about from a recursive construction of each real normed division algebra – starting from $\mathbb{C} := \mathbb{R} + \mathbb{R}i$, then $\mathbb{H} := \mathbb{C} + \mathbb{C}j$, and then taking a new imaginary l such that $l^2 = -1$, with i, j, k, l linearly independent, and setting $\mathbb{O} := \mathbb{H} + \mathbb{H}l$.

Definition 3 (Division algebra)**.** A division algebra is an algebra A with no zero divisors (nonzero elements $a, b \in A$ such that $ab = 0$) [1, p. 149]:

$$\forall a, b \in A, ab = 0 \implies a = 0 \text{ or } b = 0$$

Definition 4 (Normed division algebra [1, p. 149])**.** A normed division algebra is an algebra A which is also a normed vector space $(A, \|\cdot\|)$ wherein the composition law holds [2, p. 68]:

$$\forall a, b \in A, \|ab\| = \|a\|\|b\|$$

Remark. Note that this definition doesn't specify that A is a division algebra, as this property is implied: using separation of points for norms ($\|x\| = 0 \iff x = 0$), we get

$$\begin{aligned} ab = 0 &\implies \|a\|\|b\| = \|ab\| = \|0\| = 0 \\ &\implies \|a\| = 0 \text{ or } \|b\| = 0 \\ &\implies a = 0 \text{ or } b = 0 \end{aligned}$$

There are only four real normed division algebras (that is, normed division algebras over $\mathbb{R}$) containing the multiplicative identity 1, which are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ [1, p. 150]. For these, there is a standard definition for the norm $\|\cdot\|$, which requires a little more groundwork:

Definition 5 (Conjugate)**.** For $a \in \mathbb{R}$,

$$a^* = a$$

For $a + bi \in \mathbb{C}$ $(a, b \in \mathbb{R})$,

$$(a + bi)^* = a - bi$$

For $a + bi + cj + dk \in \mathbb{H}$ $(a, b, c, d \in \mathbb{R})$ [9, p. 232],

$$(a + bi + cj + dk)^* = a - bi - cj - dk$$

For $a_0 + \sum_{k=1}^{7} a_k e_k \in \mathbb{O}$ $(a_0, \ldots, a_7 \in \mathbb{R})$ [9, p. 237],

$$\left(a_0 + \sum_{k=1}^{7} a_k e_k\right)^* = a_0 - \sum_{k=1}^{7} a_k e_k$$

Using this conjugate, we define the norm by $\|a\|^2 = aa^*$ [1, p. 154] so, in keeping with non-negativity of norms, $\|a\| = \sqrt{aa^*}$. This agrees with our usual definition for the modulus $|\cdot|$ of real and complex numbers, since (for $a, b \in \mathbb{R}$),

$$\begin{aligned} \|a\|^2 &= a^2 \\ \|a + bi\|^2 &= a^2 + b^2 \end{aligned}$$

In fact, we can show by explicit calculation that the norms for quaternions and octonions are of a similar form, i.e. sums of four and eight squares respectively (multiplication of octonions will be properly defined later):

$$\begin{aligned}\|a+bi+cj+dk\|^2 &= (a+bi+cj+dk)(a-(bi+cj+dk))\\ &= a^2-(bi+cj+dk)^2\\ &= a^2-b^2i^2-c^2j^2-d^2k^2-bc(ij+ji)-cd(jk+kj)-bd(ik+ki)\\ &= a^2+b^2+c^2+d^2\end{aligned}$$

$$\begin{aligned}\left\|a_0+\sum_{k=1}^{7}a_ke_k\right\|^2 &= \left(a_0+\sum_{k=1}^{7}a_ke_k\right)\left(a_0-\sum_{k=1}^{7}a_ke_k\right)\\ &= a_0^2-\sum_{i=1}^{7}\sum_{j=1}^{7}a_ia_je_ie_j\\ &= a_0^2-\sum_{k=1}^{7}a_k^2e_k^2\\ &= \sum_{k=0}^{7}a_k^2\end{aligned}$$

(In the sum $\sum_{i=1}^{7}\sum_{j=1}^{7}a_ia_je_ie_j$, we have $a_ia_je_ie_j=-a_ja_ie_je_i$ as will be shown later, so all terms with $i\neq j$ cancel, leaving only the terms with $i=j$.)

Definition 6 (Real and imaginary part)**.** For $a\in A$, where A is a real normed division algebra, we define

$$\text{real part of } a\text{: } \operatorname{Re}(a)=\frac{1}{2}(a+a^*)$$

$$\text{imaginary part of } a\text{: } \operatorname{Im}(a)=\frac{1}{2}(a-a^*)$$

We use the same notation to denote sets of imaginary numbers:

$$\begin{aligned}\text{imaginary complex numbers: } \operatorname{Im}(\mathbb{C}) &= \{bi : b\in\mathbb{R}\}\\ \text{imaginary quaternions: } \operatorname{Im}(\mathbb{H}) &= \{bi+cj+dk : b,c,d\in\mathbb{R}\}\\ \text{imaginary octonions: } \operatorname{Im}(\mathbb{O}) &= \left\{\sum_{k=1}^{7}a_ke_k : a_1,\dots,a_7\in\mathbb{R}\right\}\end{aligned}$$

Remark. Note that the definition of an imaginary part conflicts with what we're used to for complex numbers – if we take $a,b\in\mathbb{R}$, then $\operatorname{Im}(a+bi)$ is usually taken to be b, but using the above definition gives the following:

$$\operatorname{Im}(a+bi)=\frac{1}{2}(a+bi-(a-bi))=bi$$

Our usual convention for complex numbers cannot be used for quaternions or octonions, as the imaginary part cannot be expressed by a single real number, so we change this to $\operatorname{Im}(a+bi)=bi$ for consistency across all four algebras.

2.2 Multiplication of octonions [1, pp. 150-151]

We have multiplication of the standard basis given by the following Cayley table:

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

There are 480 permutations of $e_1, \ldots, e_7$, giving 480 possible Cayley tables. I've stuck with the convention where $e_1 e_2 = e_4$ (this equation can be used to generate the whole table, as will be shown later).

For now, we need to ensure that this table gives us all the information we need to multiply together any two octonions. From the definition of an algebra, we know the product is bilinear, meaning that for $a, b, c \in \mathbb{O}; r_1, r_2 \in \mathbb{R}$:

$$(r_1 a + r_2 b)c = r_1 ac + r_2 bc$$
$$c(r_1 a + r_2 b) = r_1 ca + r_2 cb$$

In other words, multiplication of octonions is distributive over multiplication by reals and addition. We can use this fact to multiply together octonions – if we let $a_0, \ldots, a_7, b_0, \ldots, b_7 \in \mathbb{R}$, and $a = a_0 + \sum_{k=1}^{7} a_k e_k$, $b = b_0 + \sum_{k=1}^{7} b_k e_k$, then

$$\begin{aligned} ab &= \left(a_0 + \sum_{k=1}^{7} a_k e_k\right)\left(b_0 + \sum_{k=1}^{7} b_k e_k\right) \\ &= a_0 b_0 + a_0 \sum_{k=1}^{7} b_k e_k + b_0 \sum_{k=1}^{7} a_k e_k + \left(\sum_{k=1}^{7} a_k e_k\right)\left(\sum_{k=1}^{7} b_k e_k\right) \\ &= a_0 b_0 + \sum_{k=1}^{7} (a_0 b_k + b_0 a_k) e_k + \sum_{i=1}^{7} \sum_{j=1}^{7} a_i b_j e_i e_j \end{aligned}$$

You can then work out each $e_i e_j$ using the table, and hence find the product of any two octonions.

However, the table is fairly unwieldly, and nigh impossible to remember. It helps to find some way to summarise the rules for octonion multiplication, just as the rules for quaternion multiplication can be summarised in one line – unfortunately, it isn't quite as simple.

3 Constructing the Cayley table

There are several approaches to formulating rules for multiplication – I've found that the two sets of rules below work well:

3.1 Index cycling and doubling [1, p. 151]

Starting from $e_1e_2 = e_4$, the entire Cayley table in the previous section can be generated using the following rules (thinking of the indices i, j, k as elements of $\mathbb{Z}/7\mathbb{Z} = \{1, \ldots, 7\}$).

$$\begin{aligned} e_k^2 &= -1 \\ i \neq j &\implies e_ie_j = -e_je_i && \text{(anticommutativity)} \\ e_ie_j = e_k &\implies e_{i+1}e_{j+1} = e_{k+1} && \text{(index cycling)} \\ e_ie_j = e_k &\implies e_{2i}e_{2j} = e_{2k} && \text{(index doubling)} \end{aligned}$$

3.2 Quaternion triplets [2, p. 75]

As stated before, the rules for multiplication for $i, j, k \in \mathbb{H}$ are

$$i^2 = j^2 = k^2 = ijk = -1$$

and from this, you can work out how to multiply all of the quaternions.

You can apply a similar process using the so-called "quaternion triplets"

$$(i, j, k) = (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 7), (5, 6, 1), (6, 7, 2), (7, 1, 3)$$

for which $e_i, e_j, e_k \in \mathbb{O}$ behave similarly to $i, j, k \in \mathbb{H}$, i.e.

$$e_i^2 = e_j^2 = e_k^2 = (e_ie_j)e_k = e_i(e_je_k) = -1$$

However, this construction isn't enough by itself – octonions are non-associative, so we don't know how to get the value of e_ie_j out of $(e_ie_j)e_k = -1$, since we can't do much with $((e_ie_j)e_k)e_k = -e_k$. We need alternativity, which allows us to turn this into $e_ie_j = e_k$; and for other octonions that go beyond the basis vectors, we need to prove even more properties to avoid gruelling calculations.

4 Properties of Octonions

> "$A \cdot BC = AB \cdot C = ABC$ if A, B, C be quaternions, but not so, generally, with your octaves." – Hamilton, July 1844, in a letter to Graves [2, p. 9]

Though the octonions are non-associative, they satisfy a weaker version of this property, known as alternativity.

Proposition 1 (Alternative [1, pp. 149-150])**.** *$\mathbb{O}$ is alternative – one definition of this is that $\forall a, b \in \mathbb{O}$:*

$$a(ab) = (aa)b$$
$$a(bb) = (ab)b$$

Corollary 1.1 (Flexible)**.** $\forall a, b \in \mathbb{O}, a(ba) = (ab)a$

Proof. The following proof is elaborated from the sketch proof given by Robert Wisbauer for alternative algebras in general [10, p. 14]:

Let $c \in \mathbb{O}$ and consider $(a+b)^2c$. Using distributivity and alternativity of octonionic multiplication, we get the following:

$$\begin{aligned}(aa)c + (ab)c + (ba)c + (bb)c &= (a^2 + ab + ba + b^2)c \\ &= ((a+b)(a+b))c \\ &= (a+b)((a+b)c) \\ &= (a+b)(ac+bc) \\ &= a(ac) + b(ac) + a(bc) + b(bc) \\ &= (aa)c + b(ac) + a(bc) + (bb)c\end{aligned}$$

We can subtract $(aa)c + (bb)c$ from both sides, and then set $c = a$:

$$\begin{aligned}(ab)c + (ba)c &= b(ac) + a(bc) \\ (ab)a + (ba)a &= b(aa) + a(ba) && \text{(setting } c = a\text{)} \\ &= (ba)a + a(ba) && \text{(alternativity)} \\ (ab)a &= a(ba)\end{aligned}$$

□

Proposition 2 (Alternativity extended [6, p. 18] (4.17))**.** *As Tevian Dray put it, "alternativity extends to products with conjugates":*

$$a(a^*b) = (aa^*)b = \|a\|^2 b$$
$$(ba)a^* = b(aa^*) = \|a\|^2 b$$

Except for the cases above, we generally find $a(bc) \neq (ab)c$ for $a, b, c \in \mathbb{O}$. In fact, John Conway describes the octonions as "strongly non-associative" [2, p. 89], in that they satisfy the following property:

Lemma 3. *Let $r \in \mathbb{O}$. Then*

$$\forall x, y \in \mathbb{O}, x(ry) = (xr)y \iff r \in \mathbb{R}$$

Proof. If we start with $r \in \mathbb{R}$, then multiplying by r is simply scalar multiplication (when considering $\mathbb{O}$ as a vector space over $\mathbb{R}$) – this is commutative and associative, so for any octonions x, y, we have $x(ry) = r(xy) = (rx)y = (xr)y$.

For the other direction, I have adapted the following proof from [2, p. 90] (changing notation for consistency and elaborating on a couple of steps):

Let $r = r_0 + \sum_{k=1}^{7} r_k e_k$, with $r_0, \ldots, r_7 \in \mathbb{R}$, and assume that all octonions x, y satisfy $x(ry) = (xr)y$. Then we have

$$\begin{aligned}
x(ry) &= x\left(\left(r_0 + \sum_{k=1}^{7} r_k e_k\right) y\right) \\
&= x(r_0 y + r_1 e_1 y + \cdots + r_7 e_7 y) && \text{(distributivity)} \\
&= r_0 xy + r_1 x(e_1 y) + \cdots + r_7 x(e_7 y) && \text{(distributivity)} \\
(xr)y &= r_0 xy + r_1 (xe_1)y + \cdots + r_7 (xe_7)y
\end{aligned}$$

So for $x(ry) = (xr)y$ to be true $\forall x, y \in \mathbb{O}$, we need to have $\sum_{k=1}^{7} r_k x(e_k y) = \sum_{k=1}^{7} r_k (xe_k)y$. Since $r_k \in \mathbb{R}$, this means that the following statement has to be satisfied $\forall x, y \in \mathbb{O}; k = 1, \ldots, 7$:

$$r_k x(e_k y) = r_k (xe_k)y$$

Assume that this is true for all $x, y \in \mathbb{O}$. Then we can set $k = 1, x = e_2, y = e_3$:

$$e_2(e_1 e_3) = e_2 e_7 = -e_6 = e_4 e_3 = -(e_2 e_1)e_3$$

So $r_1 e_2(e_1 e_3) = r_1 (e_2 e_1)e_3 = -r_1 e_2(e_1 e_3)$ (first equivalence from the assumption that $x(ry) = (xr)y$, second by the direct calculation above), meaning $r_1 = -r_1$, so $r_1 = 0$.

Using the index cycling identity from earlier ($e_i e_j = e_k \implies e_{i+1} e_{j+1} = e_{k+1}$) and anticommutativity ($i \neq j \implies e_i e_j = -e_j e_i$) to deal with the case $e_i e_j = -e_k$, we can get

$$\forall n \in \mathbb{N}, e_i e_j = \pm e_k \implies e_{i+n} e_{j+n} = \pm e_{k+n}$$

We then use this on the equation $e_2(e_1 e_3) = -(e_2 e_1)e_3$ to get

$$e_{2+n}(e_{1+n} e_{3+n}) = -(e_{2+n} e_{1+n})e_{3+n}$$

meaning that for each $e_k, k = 1, \ldots, 7$, we can find a pair of octonions x, y that gives $x(e_k y) = -(xe_k)y$. As above, $r_k = -r_k$, so $r_k = 0$ for $k = 2, \ldots, 7$. Thus,

$$r = r_0 + \sum_{k=1}^{7} 0 e_k = r_0 \in \mathbb{R}$$

□

Proposition 4 (Conjugate of product [6, p. 17](4.9))**.** *For $a, b \in \mathbb{O}$,*

$$(ab)^* = b^* a^*$$

Remark. This is also true for any other real normed division algebra.

Proposition 5 (Inverse [6, p. 17](4.11))**.** *Let $a \in \mathbb{O} \setminus \{0\}$. Given the norm defined by $\|a\|^2 = aa^* = a^*a$, and that the multiplicative inverse of a is a^{-1} such that $aa^{-1} = a^{-1}a = 1$, we have*

$$a^{-1} = \frac{a^*}{\|a\|^2}$$

Proof. This can easily be verified:

$$a\left(\frac{a^*}{\|a\|^2}\right) = \frac{aa^*}{\|a\|^2} = \frac{\|a\|^2}{\|a\|^2} = 1$$
$$\left(\frac{a^*}{\|a\|^2}\right)a = \frac{a^*a}{\|a\|^2} = \frac{\|a\|^2}{\|a\|^2} = 1$$

□

Remark. Again, this is true for any other real normed division algebra.

Proposition 6 (Latin square property [11, p. 45])**.** *The nonzero octonions satisfy the Latin square property – that is, if $a, b \in \mathbb{O} \setminus \{0\}$, then $\exists! x, y \in \mathbb{O}$ such that $ax = b = ya$.*

Proof. For existence, we can take $x = a^{-1}b$ and $y = ba^{-1}$:

$$\begin{aligned}
ax &= \frac{a(a^*b)}{\|a\|^2} \\
&= \frac{(aa^*)b}{\|a\|^2} && \text{(extended alternativity)} \\
&= b && (aa^* = \|a\|^2) \\
ya &= \frac{(ba^*)a}{\|a\|^2} \\
&= \frac{b(a^*a)}{\|a\|^2} && \text{(extended alternativity)} \\
&= b && (a^*a = \|a\|^2)
\end{aligned}$$

For uniqueness, say that $\exists x, x' \in \mathbb{O}$ such that $ax = ax' = b$. Then

$$0 = ax - ax' = a(x - x') \text{ (distributivity)}$$

Since $\mathbb{O}$ is a division algebra, $a(x - x') = 0, a \neq 0 \implies x - x' = 0$, so $x = x'$. Analogously, if $\exists y, y' \in \mathbb{O}$ such that $ya = y'a = b$, then $(y - y')a = 0$, so $y = y'$.

Hence, x and y are the only octonions that satisfy $ax = b = ya$. □

Remark. A Latin square is a matrix wherein every element appears exactly once in each row and column - since this property is satisfied, the Cayley table for the non-zero octonions will be an (infinitely large) Latin square.

Also, if this property is assumed to be true, then it implies that there are no zero divisors, meaning that a division algebra can also be defined as an algebra satisfying the Latin square property. [12, p. 51]

We can use the Latin square property and inverse to finally define a notion of division on our division algebra $\mathbb{O}$. It's important to note that this isn't a single operation, as with real numbers and complex numbers - we can't define $\frac{a}{b}$ for octonions, or even for quaternions, since this requires commutativity. However, we can define two operations:

Definition 7 (Left and right division)**.** The uniqueness in the Latin square property allows us to define left \ and right / division:

$$a(a\backslash b) = b$$
$$(b/a)a = b$$

Combining this with the x and y given in the proof of the Latin square property, we can give explicit formulae for these operations as follows:

$$a\backslash b = \frac{a^*b}{\|a\|^2}$$
$$b/a = \frac{ba^*}{\|a\|^2}$$

Proposition 7 (Anticommutative [6, p. 20](4.37))**.** *For $a, b \in Im(\mathbb{O})$ - that is, imaginary octonions, so where the real part is 0 - we have*

$$ab = -ba$$

Proof. We know anticommutativity holds for distinct e_i, e_j, i.e. $e_ie_j = -e_je_i$. If we let $a = \sum_{k=1}^7 a_ke_k, b = \sum_{k=1}^7 b_ke_k$, with $a_k, b_k \in \mathbb{R}$, then

$$\begin{aligned} ab &= \sum_{i=1}^7 \sum_{j=1}^7 a_ib_je_ie_j \\ &= \sum_{i=1}^7 \sum_{j=1}^7 a_ib_j(-e_je_i) && \text{(anticommutativity)} \\ &= -\sum_{j=1}^7 \sum_{i=1}^7 b_ja_ie_je_i \\ &= -ba \end{aligned}$$

□

Lemma 8 ($\sqrt{-1}$)**.** *In $\mathbb{O}$, there are infinitely many square roots of -1.*

The square roots of -1 aren't limited to $\pm e_k$; in fact, for any $a \in \text{Im}(\mathbb{O})$, all you need for $a^2 = -1$ to hold is $\|a\| = 1$:

$$\left(\sum_{k=1}^7 a_ke_k\right)^2 = \sum_{k=1}^7 a_k^2e_k^2 = -\sum_{k=1}^7 a_k^2 = -\|a\|$$

This works in other real normed division algebras as well (so there are no square roots of -1 in $\mathbb{R}$, since the imaginary part is always 0; the only square roots in $\mathbb{C}$ are $\pm i$; and in $\mathbb{H}$ you have any $bi + cj + dk$ satisfying $b^2 + c^2 + d^2 = 1$).

5 Uses of octonions

Many of the uses of octonions require a lot of background knowledge in other areas of mathematics or physics, and as such cannot be included here. However, there are a couple of applications that follow from the introduction above.

There is an interesting observation to be made with both of the examples presented here: they don't involve anything to do with division algebras in their formulation, but the octonions are still behind the scenes, lending some structure to what seems overly convoluted at first glance.

5.1 Seven-dimensional cross product

Definition 8 (Dot product [13, p. 698])**.** The dot product on $\mathbb{R}^n$ is a map taking $x, y \in \mathbb{R}^n$ to $x \cdot y \in \mathbb{R}$, defined by

$$x \cdot y = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum_{k=1}^{n} x_k y_k$$

Definition 9 (Euclidean norm [13, p. 698])**.** For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{k=1}^{n} x_k^2}$$

Definition 10 (Cross product [13, p. 698])**.** A cross product is a map taking $x, y \in \mathbb{R}^n$ to $x \times y \in \mathbb{R}^n$ satisfying the following conditions for $\alpha, \beta \in \mathbb{R}; x, y, z \in \mathbb{R}^n$:

$$\begin{aligned} \text{bilinearity: } (\alpha x + \beta y) \times z &= \alpha(x \times z) + \beta(y \times z) \\ z \times (\alpha x + \beta y) &= \alpha(z \times x) + \beta(z \times y) \\ \text{orthogonality: } x \cdot (x \times y) &= (x \times y) \cdot y = 0 \\ \text{magnitude: } |x \times y|^2 &= |x|^2 |y|^2 - (x \cdot y)^2 \end{aligned}$$

Cross products on $\mathbb{R}^n$ only exist for $n = 0, 1, 3, 7$ (here $\mathbb{R}^0 = \{0\} = \mathrm{Im}(\mathbb{R})$), corresponding to $\mathrm{Im}(A)$ for real normed division algebras A [6, pp. 20-21] – and we can use a similar sort of process to find a cross product in each case.[3]

If we have $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then we consider these as elements $a, b \in \mathrm{Im}(A)$:

case	a	b
$\mathbf{a}, \mathbf{b} \in \mathbb{R}$, so $a, b \in \mathrm{Im}(\mathbb{C})$	$a_1 i$	$b_1 i$
$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, so $a, b \in \mathrm{Im}(\mathbb{H})$	$a_1 i + a_2 j + a_3 k$	$b_1 i + b_2 j + b_3 k$
$\mathbf{a}, \mathbf{b} \in \mathbb{R}^7$, so $a, b \in \mathrm{Im}(\mathbb{O})$	$\sum_{k=1}^{7} a_k e_k$	$\sum_{k=1}^{7} b_k e_k$

[3]Both sources used for this subsection ([13], [6]) state that cross products only exist for $n = 3, 7$, and don't include the cases $n = 0, 1$; a little working from the definition above shows that the zero map is technically a cross product on $\mathbb{R}$ (and in fact, the only possibility), and it works trivially for $n = 0$. The zero map fails the magnitude condition for $n > 1$.

Then a cross product can be calculated using complex/quaternionic/octonionic multiplication, restricted to the imaginary part (i.e. you calculate $\mathrm{Im}(ab)$).

In $\mathbb{R}^3$, we've learnt the formula for the cross product as follows:

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

This corresponds to the above definition – taking $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k \in \mathrm{Im}(\mathbb{H})$, we get the following [6, pp. 20–21]:

$$\begin{aligned}\mathrm{Im}(ab) &= \mathrm{Im}(-(a_1b_1 + a_2b_2 + a_3b_3) + (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k) \\ &= (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k\end{aligned}$$

The case $n = 7$ is unfamiliar, but it is straightforward enough to find using the same method. Let $\mathbf{a} = (a_1, \ldots, a_7), \mathbf{b} = (b_1, \ldots, b_7) \in \mathbb{R}^7$ correspond to $a = \sum_{k=1}^{7} a_ke_k$ and $b = \sum_{k=1}^{7} b_ke_k$ respectively. We can then use octonionic multiplication restricted to $\mathrm{Im}(\mathbb{O})$ to define a cross product $\mathbf{a} \times \mathbf{b}$:

$$\begin{aligned}\mathrm{Im}(ab) &= ab - \mathrm{Re}(ab) \\ &= \sum_{i=1}^{7}\sum_{j=1}^{7} a_ib_je_ie_j + \sum_{k=1}^{7} a_kb_k \\ &= \sum_{k=1}^{7} x_ke_k\end{aligned}$$

where $x_1, \ldots, x_7 \in \mathbb{R}$ are found by explicit calculation. Then the following is a cross product in $\mathbb{R}^7$ [6, pp. 20–21]:

$$\mathbf{a} \times \mathbf{b} = (x_1, \ldots, x_7)$$

Remark. The above cross product in $\mathbb{R}^7$ is anticommutative (just like in $\mathbb{R}^3$):

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

This ties in to the fact that imaginary octonions anticommute ($ab = -ba$).

5.2 Eight squares theorem

In short, this theorem states that the product of two numbers that are each the sum of eight squares is also a sum of eight squares.

No one mathematician can be credited with its discovery: although C. F. Degen published this theorem in 1818, he failed to specify some of the signs. Then around 1844, Graves, Cayley, and J. R. Young all came up with the same formulation of the theorem (up to notation) [14, pp. 164–165], which is recorded in Volume III of the Proceedings of the Royal Irish Academy as follows [8, p. 527]:

$$
\begin{aligned}
(s'^2 + t'^2 + u'^2 + v'^2 + w'^2 + x'^2 + y'^2 + z'^2)&(s^2 + t^2 + u^2 + v^2 + w^2 + x^2 + y^2 + z^2)\\
&= (ss' + tt' + uu' + vv' + ww' + xx' + yy' + zz')^2\\
&+ (st' - ts' + uv' - vu' + wx' - xw' + yz' - zy')^2\\
&+ (su' - us' + vt' - tv' + yw' - wy' + xz' - zx')^2\\
&+ (sv' - vs' + tu' - ut' + wz' - zw' + xy' - yx')^2\\
&+ (sw' - ws' + xt' - tx' + uy' - yu' + zv' - vz')^2\\
&+ (sx' - xs' + tw' - wt' + yv' - vy' + zu' - uz')^2\\
&+ (sy' - ys' + zt' - tz' + vx' - xv' + wu' - uw')^2\\
&+ (sz' - zs' + ty' - yt' + vw' - wv' + ux' - xu')^2
\end{aligned}
$$

Here we have an equation with far too many terms to remember and an unclear structure, but as with the seven-dimensional cross product, we can use octonions to make some sense of it.

If $a_0, \ldots, a_7 \in \mathbb{Z}$, then taking $a = a_0 + \sum_{k=1}^{7} a_k e_k \in \mathbb{O}$ and using the definition for the norm $\|\cdot\|$ on $\mathbb{O}$, we get

$$\|a\|^2 = \sum_{k=0}^{7} a_k^2$$

so $\|a\|^2$ is the sum of eight square numbers.

The eight squares theorem is equivalent to the composition law for octonions with integer coefficients, i.e. for $a, b \in \mathbb{O}$, $\|a\|^2\|b\|^2 = \|ab\|^2$: $\|a\|^2\|b\|^2$ is a product of two sums of eight squares, and $\|ab\|^2$ is a sum of eight squares, so using octonionic multiplication to explicitly calculate the product ab gives the above formulation (up to changes in notation) – although it is still fairly complicated to set out rules for multiplying octonions, it gives a far simpler way of stating this theorem.

Analogous theorems exist corresponding to the other three real normed division algebras – using $\|x\|^2\|y\|^2 = \|xy\|^2$ for $x, y \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ gives the following theorems (where $a, b, \ldots, h \in \mathbb{R}$) [2, p. 77]:

- For $a, b \in \mathbb{R}$:

$$
\begin{aligned}
a^2 b^2 &= \|a\|^2\|b\|^2\\
&= \|ab\|^2\\
&= (ab)^2
\end{aligned}
$$

- For $a + bi, c + di \in \mathbb{C}$:

$$
\begin{aligned}
(a^2 + b^2)(c^2 + d^2) &= \|a + bi\|^2\|c + di\|^2\\
&= \|(a + bi)(c + di)\|^2\\
&= (ac - bd)^2 + (ad + bc)^2
\end{aligned}
$$

- For $a+bi+cj+dk, e+fi+gj+hk \in \mathbb{H}$:

$$\begin{aligned}(a^2+b^2+c^2+d^2)(e^2+f^2+g^2+h^2)& \\ &= \|a+bi+cj+dk\|^2\|e+fi+gj+hk\|^2 \\ &= \|(a+bi+cj+dk)(e+fi+gj+hk)\|^2 \\ &= (ae-bf-cg-dh)^2+(af+be+ch+dg)^2 \\ &\quad+(ag-bh+ce+df)^2+(ah+bg+cf+de)^2\end{aligned}$$

5.2.1 Generalisation

In general, an N-square identity is

$$(x_1^2+\cdots+x_N^2)(y_1^2+\cdots+y_N^2)=z_1^2+\cdots+z_N^2$$

where $z_1,\ldots,z_N$ are functions of $x_1,\ldots,x_N,y_1,\ldots,y_N$ in the form

$$z_k=\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}x_iy_j \text{ such that } a_{ij}\in\{-1,0,1\}$$

There is a direct link between the values of N for which these identities exist and the dimensions of normed division algebras. Clearly if there were a normed division algebra with dimension $N \notin \{1,2,4,8\}$, then there would be a corresponding N-square identity by writing out the composition law in full; and if there were an N-square identity with $N \notin \{1,2,4,8\}$, it could be used to find a composition law for an N-dimensional real normed division algebra.

6 Conclusion

The octonions came about as part of an exploration into working out how many 'imaginaries' could be invented, as opposed to trying to solve an existing problem – they weren't as useful as existing number systems from the outset, and still have a reputation of being fairly odd and unused. There are several applications beyond the scope of this essay: the study of octonions has been combined with other areas, including projective geometry, number theory and integers (extending the concept of Gaussian integers), eigenvalue problems, and topology [6]. They also come up in theoretical physics, especially supersymmetric string theory, but most of us could get by without ever knowing of them.

However, they are still interesting in terms of seeing just how far you can extend the idea of complex numbers, and working with a structure that doesn't behave like most of the ones we encounter. Some have high hopes – Tevian Dray believes that the octonions "will ultimately be seen as the key to understanding the basic building blocks of nature"[6, p. 2] – but for now, though indispensable for understanding some higher-dimensional problems, the octonions remain a mathematical curiosity, and as John Baez eloquently puts it:

> "Octonions rock!" [15]

References

[1] John C Baez. The Octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205, 2002.

[2] John H Conway and Derek A Smith. On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry. *Bulletin of the American Mathematical Society*, 42:229–243, 2005.

[3] Orlando Merino. A Short History of Complex Numbers. *University of Rhode Island*, 2006.

[4] Paul J Nahin. *An Imaginary Tale: The Story of* $\sqrt{-1}$. Princeton University Press, 2010.

[5] William Dunham. *Euler: The Master of Us All*. Number 22. MAA, 1999.

[6] Tevian Dray and Corinne A Manogue. *The Geometry of the Octonions*. World Scientific, 2015.

[7] John C Baez and John Huerta. The Strangest Numbers in String Theory. *Scientific American*, 304(5):60–65, 2011.

[8] *Proceedings of the Royal Irish Academy*, volume 3. Gill, M. H., 1847.

[9] John C Baez. On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry by John H. Conway and Derek A. Smith. *Bull. Amer. Math. Soc*, 42:229–243, 2005.

[10] Robert Wisbauer. *Modules and Algebras: Bimodule Structure on Group Actions and Algebras*, volume 81. CRC Press, 1996.

[11] C. R. Jordan and D. A. Jordan. *Groups*. Arnold, E., 1994.

[12] Susumo Okubo. *Introduction to Octonion and Other Non-associative Algebras in Physics*, volume 2. Cambridge University Press, 1995.

[13] WS Massey. Cross Products of Vectors in Higher Dimensional Euclidean Spaces. *The American Mathematical Monthly*, 90(10):697–701, 1983.

[14] Leonard E Dickson. On Quaternions and Their Generalization and the History of the Eight Square Theorem. *Annals of Mathematics*, pages 155–171, 1919.

[15] John C Baez. This Week's Finds in Mathematical Physics (Week 260), 2009.